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Bianchi Type III Cosmological Models for Barotropic Perfect Fluid Distribution with Variable G and Λ

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Abstract Bianchi type III cosmological model for perfect fluid distribution with variable G and Λ are investigated. To get the determinate models, we have assumed the barotropic condition $p = \gamma\rho$ and shear (σ) is proportional to expansion (θ) where p is isotropic pressure, ρ the matter density and $0 \le \gamma \le 1$. The physical and geometrical aspects related with the observations and singularities in the models are discussed.

Keywords Bianchi type III · Cosmological · Barotropic · Variable G, Λ

1 Introduction

In general relativity, spatially homogeneous space-times either belong to Bianchi type or Kantowski-Sachs space-times and interpreted as cosmological models (Raychaudhuri [1]). Friedmann-Robertson-Walker models are unstable near the singularity (Patridge and Wilkinson [2]) and fail to describe early universe. Therefore, spatially homogeneous and anisotropic Bianchi space-times are undertaken to study the universe in its early stages of evolution. Recent cosmological observations support the existence of anisotropic universe that approaches to isotropic phase (Land and Magueijo [3]). The gravitational constant G plays the role of coupling constant between geometry and matter in Einstein's field equations. It is therefore natural to look at this constant as a function of time in an evolving universe.

Subsequently, many well posed gravitation theories were developed to generalize Einstein's general relativity by including variable G satisfying conservation equations. The first

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scalar tensor theory of gravity was formulated by Jordan [4] and then Brans and Dicke [5]. The extension of Einstein's general theory of relativity has also been proposed in order to achieve a possible unification of gravitation and elementary particle physics or to incorporate Mach's principle in general relativity (Hoyle and Narlikar [6, 7]).

In the early universe, the non-trivial role of vacuum generates a Λ(cosmological constant) term in Einstein's field equation which leads to the inflationary scenario (Abers and Lee [8]) which predicted that during an early exponential phase, the vacuum energy is treated as large cosmological constant which is expected by Glashow-Salam-Weinberg model [9] and by Grand unified theory (GUT) [10]. Therefore, the present day observations of smallness of cosmological constant ($\Lambda_0 \leq 10^{-56}$ cm^{-2}) support to assume that the cosmological constant Λ is time dependent. Gibbons and Hawking [11] conjectured that any cosmology with a positive cosmological constant Λ would asymptotically approach to a de-Sitter space-time. Therefore, cosmological models linking the variation of G with that of cosmological constant leaving the form of Einstein's field equations unchanged and preserving the conservation of energy-momentum tensor of matter content have been investigated. Beesham [12], Abdel-Rahman [13], Berman [14], Abdussattar and Vishwakarma [15] have investigated cosmological models with variable G and Λ, considering FRW space-times. Beesham [16] obtained anisotropic solution in Bianchi type I model where Λ varies as inverse of square of time. Kalligas et al. [17] investigated the solution considering Bianchi type I space-time assuming energy density $\rho \propto t^n$, $G \propto t^m$ where m, n are constants. Several cosmological models in which Λ decays with time have been investigated by number of authors viz. Abdussattar and Vishwakarma [18], Pradhan et al. [19–21], Singh and Chaubey [22], Bali et al. [23–25], Singh and Baghel [26]. Recently Singh and Kale [27] have discussed anisotropic Bianchi Type V bulk viscous cosmological models with variable G and Λ.

In this paper, we have investigated Bianchi type III cosmological models for barotropic perfect fluid distribution with time dependent G and Λ. To get the deterministic solutions, we have also assumed that the shear (σ) is proportional to expansion (θ). The physical and geometrical aspects related with the observations and singularities in the models are discussed.

2 The Metric and Field Equations

We consider the Bianchi Type III metric in the form

$$dS^2 = -dt^2 + A^2dx^2 + B^2e^{2x}dy^2 + C^2dz^2 \tag{1}$$

where A, B and C are functions of t-alone.

The Einstein's field equations for time dependent G and Λ are given by

$$R_i^j - \frac{1}{2}Rg_i^j = -8\pi GT_i^j + \Lambda g_i^j \tag{2}$$

where the energy-momentum tensor T_i^j for perfect fluid distribution is given by

$$T_i^j = (\rho + p)v_iv^j + pg_i^j \tag{3}$$

where p is the isotropic pressure, ρ the energy density and v^i the flow vector satisfying $v^1 = 0 = v^2 = v^3$, $v_iv^i = -1$.

Now the Einstein's field equation (2) for the metric (1) leads to

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}=-8\pi Gp+\Lambda, \tag{4}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}=-8\pi Gp+\Lambda, \tag{5}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}-\frac{1}{A^2}=-8\pi Gp+\Lambda, \tag{6}$$

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{A}\dot{C}}{AC}-\frac{1}{A^2}=8\pi G\rho+\Lambda \tag{7}$$

and

$$\frac{\dot{A}}{A}-\frac{\dot{B}}{B}=0 \tag{8}$$

where the overdot denotes differentiation with respect to time t. An additional equation for time changes of G and Λ are obtained by the divergence of Einstein's tensor i.e. $(R_i^j-\frac{1}{2}Rg_i^j)_{;j}=0$ which leads to

$$(8\pi GT_i^j-\Lambda g_i^j)_{;j}=0 \tag{9}$$

which yields

$$8\pi\dot{G}\rho+\dot{\Lambda}+8\pi G\left[\dot{\rho}+(\rho+p)\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\right]=0. \tag{10}$$

From equation (10), we have

$$\dot{\rho}+(\rho+p)\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)=0 \tag{11}$$

and

$$8\pi\dot{G}\rho+\dot{\Lambda}=0. \tag{12}$$

There are five equations (4)–(8) in seven unknowns A, B, C, p, ρ, G and Λ. To find the complete solution of the system, we assume the barotropic condition

$$p=\gamma\rho, \quad 0\le\gamma\le 1 \tag{13}$$

and shear (σ) is proportional to expansion (θ) i.e.

$$\sigma\propto\theta \tag{14}$$

which leads to

$$B=C^n. \tag{15}$$

Here p is the isotropic pressure, ρ the energy density, σ the shear, θ the expansion, B and C are metric potentials, n is a constant and proportionality constant is assumed as unity.

The motive behind assuming the condition $\sigma \propto \theta$ is explained as referring to Thorne [28], the observations of the velocity red-shift relation for extragalactic sources suggest that the Hubble expansion of the universe is isotropic today to within 30 percent (Kantowski and Sachs [29], Kristian and Sachs [30]). More precisely, the red-shift studies place the limit $\frac{\sigma}{H} \leq 0.30$, where σ is shear and H is the Hubble constant. Collins et al. [31] have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous hypersurface satisfies the condition $\frac{\sigma}{\theta}$ is constant, where σ is shear and θ the expansion in the model.

3 Solution of Field Equations

Equation (8) leads to

$$A = \alpha B, \tag{16}$$

where α is a constant. Equations (5) and (6) lead to

$$\frac{\ddot{C}}{C} - \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\left(\frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right) + \frac{1}{A^2} = 0. \tag{17}$$

From (15), (16) and (17), we have

$$\frac{\ddot{C}}{C} + 2n\frac{\dot{C}^2}{C^2} = \frac{1}{\alpha^2(n-1)C^{2n}}$$

which leads to

$$2\ddot{C} + \frac{4n}{C}\dot{C}^2 = \frac{2}{\alpha^2(n-1)}.\frac{1}{C^{2n-1}}. \tag{18}$$

Setting $\dot{C} = f(C)$ in (18), we have

$$f^2 = \left(\frac{dC}{dt}\right)^2 = \frac{1}{\alpha^2(n-1)}C^{2-2n} + \beta C^{-4n}, \tag{19}$$

β being a constant. Here, we consider two cases:

(i) $\beta = 0$,
(ii) $\beta \neq 0$.

Case (i): when $\beta = 0$ then (19) leads to

$$C^{n-1}dC = \frac{1}{\alpha\sqrt{n^2-1}}dt \tag{20}$$

thus we have

$$C^n = at + b \tag{21}$$

where $a = \frac{n}{\alpha\sqrt{n^2-1}}$ and b is the constant of integration.

Now

$$B = C^n = at + b, \tag{22}$$

and

$$A = \alpha B = \alpha(at + b). \tag{23}$$

After suitable transformation of coordinates, the metric (1) leads to

$$dS^2 = -\frac{dT^2}{a^2} + T^2 dX^2 + T^2 e^{2X/\alpha} dY^2 + T^{2/n} dZ^2, \tag{24}$$

where $at + b = T$, $\alpha x = X$, $y = Y$, $z = Z$.

Using barotropic condition $p = \gamma\rho$ and (15) and (16) in (11), we have

$$\frac{\dot{\rho}}{\rho} = -\frac{(\gamma + 1)(2n + 1)a}{n(at + b)}. \tag{25}$$

This leads to

$$\rho = \frac{N}{(at + b)^S} = \frac{N}{T^S} \quad \text{where} \tag{26}$$

$$S = \frac{(\gamma + 1)(2n + 1)}{n} \quad \text{and} \tag{27}$$

$$(at + b) = T, \tag{28}$$

N is a constant of integration. The Hubble parameter (H) is given by

$$H = \frac{(2n + 1)a}{3nT}. \tag{29}$$

The expansion (θ) and shear (σ) are given by

$$\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} = \frac{(2n + 1)a}{nT} \quad \text{and} \tag{30}$$

$$\sigma = \frac{1}{\sqrt{3}} \frac{(n - 1)}{n} \frac{a}{(at + b)} = \frac{(n - 1)a}{\sqrt{3}nT}. \tag{31}$$

The spatial volume (R^3) and deceleration parameter (q) are given by

$$R^3 = \alpha C^{2n+1/n} = \alpha T^{2n+1/n}, \tag{32}$$

$$q = -\frac{2(n - 1)}{2n + 1}. \tag{33}$$

Equations (7) and (12) lead to

$$8\pi G = \frac{2}{\alpha^2 s N} \left[\frac{a^2\alpha^2(n + 2) - n^2}{n} \right] T^{s-2}. \tag{34}$$

From (12), we have

$$\begin{aligned} \dot{\Lambda} &= -8\pi \dot{G} \rho \\ &= \frac{2a(s - 2)\{\frac{-a^2\alpha^2(n+2)+n^2}{n}\}}{\alpha^2 s T^3} \end{aligned} \tag{35}$$

which leads to

$$\Lambda = \frac{(s-2)\{a^2\alpha^2(n+2)-n^2\}}{\alpha^2 s}\cdot\frac{1}{T^2}. \tag{36}$$

Case (ii): when $\beta \neq 0$ then (19) leads to

$$\frac{dC}{dt} = \sqrt{\frac{1}{n^2-1}C^{2-2n}+\beta C^{-4n}}. \tag{37}$$

Therefore, the metric (1) reduces to the form

$$dS^2 = -\frac{d\tau^2}{\frac{1}{n^2-1}\tau^{2-2n}+\beta\tau^{-4n}} + \tau^{2n}dX^2 + \tau^{2n}e^{2X/\alpha}dY^2 + \tau^2 dZ^2 \tag{38}$$

where the cosmic time t is defined as

$$t = \int \frac{\tau^{2n}d\tau}{\sqrt{\frac{1}{n^2-1}\tau^{2+2n}+\beta}} \quad \text{and} \tag{39}$$

$$C = \tau, \qquad A = \alpha B = \alpha C^n. \tag{40}$$

Using the barotropic condition $p = \gamma\rho$ in (11), we have

$$\rho = \frac{M}{\tau^{(\gamma+1)(2n+1)}} \tag{41}$$

where M is the constant of integration.
Equations (7), (12) and (41) lead to

$$8\pi G = k\tau^{(2n+1)(\gamma-1)} + L\tau^{\gamma(2n+1)+1} \quad \text{where} \tag{42}$$

$$k = \frac{2\beta n(n+2)}{N(\gamma+1)} \quad \text{and} \tag{43}$$

$$L = \frac{2n^2\alpha^2(n+2)-2n(n^2-1)}{N\alpha^2(n^2-1)(\gamma+1)(2n+1)}. \tag{44}$$

Equations (7) and (12) lead to

$$\Lambda' = -8\pi G'\rho \quad \text{where} \tag{45}$$

$$\Lambda' = \frac{d\Lambda}{d\tau} \quad \text{and} \quad G' = \frac{dG}{d\tau}. \tag{46}$$

From (45), we have

$$\Lambda = \frac{N}{2\tau^2}\left[\frac{k(\gamma-1)}{\tau^{4n}} + \frac{L\{\gamma(2n+1)+1\}}{n\tau^{2n-2}}\right]. \tag{47}$$

The expansion (θ), shear (σ), Hubble parameter (H), spatial volume (R^3) and deceleration parameter (q) for the model (38) are given by

$$\theta = \frac{(2n+1)}{\sqrt{(n^2-1)}}\frac{\sqrt{\beta(n^2-1)+\tau^{2+2n}}}{\tau^{(2n+1)}}, \tag{48}$$

$$\sigma = \frac{1}{\sqrt{3}}(n-1)\frac{\sqrt{\beta(n^2-1)+\tau^{2+2n}}}{\tau^{(2n+1)}}, \tag{49}$$

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right) = \frac{1}{3}(2n+1)\frac{\sqrt{\beta(n^2-1)+\tau^{2+2n}}}{\tau^{(2n+1)}}, \tag{50}$$

$$R^3 = \alpha\tau^{2n+1}, \tag{51}$$

$$q = -\frac{2(n-1)}{2n+1}. \tag{52}$$

4 Discussion

The model (24) starts with a bigbang at $T = 0$ and the expansion in the model decreases as time increases. The reality conditions $(\rho + p) > 0$, $(\rho + 3p) > 0$ given by Ellis [32] are satisfied for $N > 0$. The matter density $\rho \to \infty$ when $T \to 0$ and $\rho \to 0$ when $T \to \infty$ provided $s > 0$. The spatial volume (R^3) increases as T increases if $(2n + 1) > 0$. The deceleration parameter $q < 0$ if $n > 1$ and $q > 0$ if $0 < n < 1$. Thus the model (24) represents an decelerating phase if $0 < n < 1$ and then accelerating phase if $n > 1$. Since $\frac{\sigma}{\theta} \neq 0$, hence anisotropy is maintained throughout. We also find that $|\frac{\dot{G}}{G}| \cong \frac{1}{T} \cong H$ and $\Lambda \propto \frac{1}{T^2}$. These results match with the results obtained by Beesham [16]. The model (24) has Cigar type singularity (Mac Callum [33]) at $T = 0$ if $n < 0$ and it has Point type singularity at $T = 0$ if $n > 0$.

The model (38) also starts with a bigbang at $\tau = 0$ if $(2n + 1) > 0$ and the expansion in the model decreases as time increases. The reality conditions $\rho + p > 0$, $\rho + 3p > 0$ are satisfied if $M > 0$. The matter density $\rho \to \infty$ when $\tau \to 0$ and $(2n + 1) > 0$ and $\rho \to 0$ when $\tau \to \infty$ and $(2n + 1) > 0$. The spatial volume (R^3) increases as τ increases and $(2n + 1) > 0$. The deceleration parameter $q > 0$ if $0 < n < 1$ and $q < 0$ as $n > 1$. Thus the model (38) represents a decelerating phase and then accelerating one. Since $\frac{\sigma}{\theta} \neq 0$, hence anisotropy is maintained throughout.

We also observe that $|\frac{\dot{G}}{G}| \cong \frac{1}{\tau} \cong H$ and $\Lambda \propto \frac{1}{\tau^2}$ in special condition. These results match with the results obtained by Beesham [16]. The model (38) has Point type singularity at $\tau = 0$ when $n > 0$ (Mac Callum [33]).

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